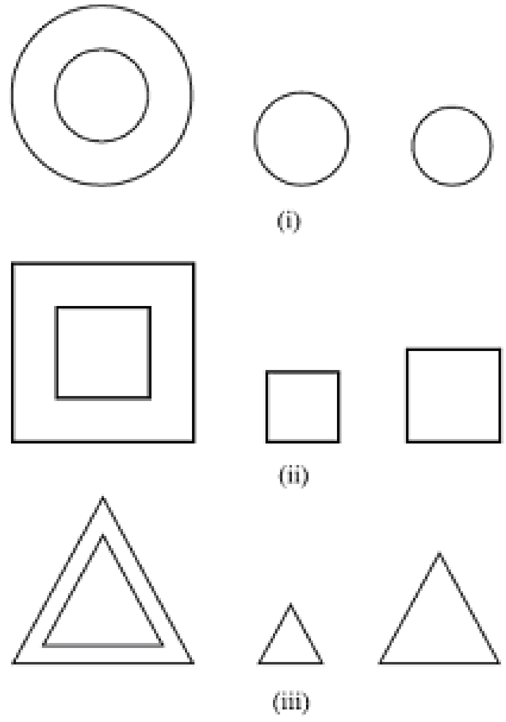
Triangles

Similar Figures



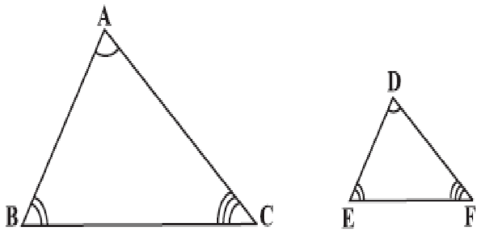
The above are similar figures.

- All congruent figures are similar but similar figures need not be congruent.
- Two regular polygons of the same number of sides are similar, if
  (i) their corresponding angles are equal and
  (ii) their corresponding sides are in the same ratio (or proportion).

Similarity of Triangles:

- Two triangles are similar, if
  a) their corresponding angles are equal and
  b) their corresponding sides are in the same ratio (or proportion).

- Criterion of similarity:
  In ΔABC and ΔDEF, if
  (i) if $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$ and
  (ii) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$, then the two triangles are similar.



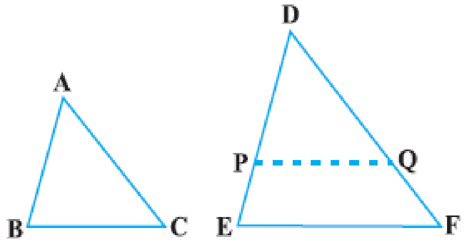
Theorems in similarity (SSS, AA, SAS, BPT)

AA criterion of similarity or AAA criterion:

If in two triangles, the corresponding angles are equal, then their corresponding sides are in the same ratio (or proportion) and hence the two triangles are similar.

This criterion is referred to as the AAA (Angle-Angle-Angle) criterion of similarity of two triangles.

This theorem can be proved by taking two triangles ABC and DEF such that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.



Cut DP = AB and DQ = AC and join PQ.

So, $\Delta ABC \cong \Delta DPQ$

This gives $\angle B = \angle P = \angle E$ and $PQ \parallel EF$

Therefore, $\frac{DP}{PE} = \frac{DQ}{QF}$

i.e. $\frac{AB}{DE} = \frac{AC}{DF}$

Similarly, $\frac{AB}{DE} = \frac{BC}{EF}$ and so

$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

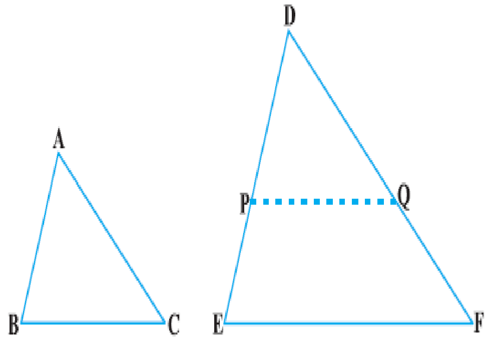
SSS criterion of similarity:

If in two triangles, the sides of one triangle are proportional to (i.e. in the same ratio of) the sides of the other triangle, then their corresponding angles are equal and hence the two triangles are similar.

This criterion is referred to as the SSS (Side-Side-Side) similarity criterion for two triangles.

This theorem can be proved by taking two triangles ABC and DEF such that

$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} (< 1)$



Cut DP = AB and DQ = AC and join PQ.

It can be seen that $\frac{DP}{PE} = \frac{DQ}{QF}$ and

PQ || EF

So, $\angle P = \angle E$ and $\angle Q = \angle F$

Therefore, $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{PQ}{EF}$

So, $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{BC}{EF}$

So, BC = PQ

SAS criterion of similarity:

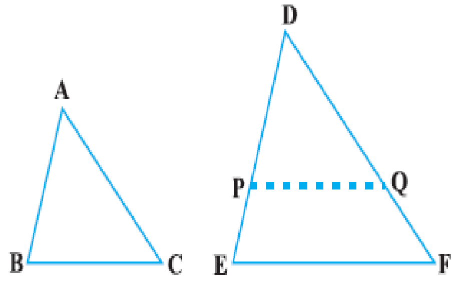
If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.

This criterion is referred to as the SAS (Side-Angle-Side) similarity criterion for two triangles.

As before, this theorem can be proved by taking two triangles ABC and DEF such that

$\frac{AB}{DE} = \frac{AC}{DF} (< 1)$ and $\angle A = \angle D$.

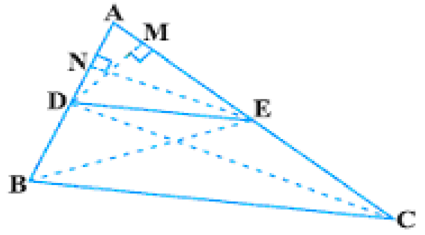
Cut DP = AB, DQ = AC and join PQ.



Now, $PQ || EF$ and $\Delta ABC \cong \Delta DPQ$
So, $\angle A = \angle D, \angle B = \angle P$ and $\angle C = \angle Q$
Therefore, $\Delta ABC \sim \Delta DEF$

Basic Proportionality theorem:
If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio.

Proof: We are given a triangle ABC in which a line parallel to side BC intersects the other two sides AB and AC at D and E respectively.



We need to prove that $\frac{AD}{DB} = \frac{AE}{EC}$

Let us join BE and CD and then draw
$DM \perp AC$ and $EN \perp AB$

Now, area of $\Delta ADE = \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times AD \times EN$

Recall from Class IX, that area of ΔADE is denoted as ar(ADE).

So, ar(ADE) $= \frac{1}{2} \times AD \times EN$

Similarly, ar(BDE) $= \frac{1}{2} \times DB \times EN$

ar(ADE) $= \frac{1}{2} \times AE \times DM$ and ar(DEC) = $\frac{1}{2} \times \text{EC} \times \text{DM}$

Therefore, $\frac{ar(ADE)}{ar(BDE)} = \frac{\frac{1}{2} \times AD \times EN}{\frac{1}{2} \times DB \times EN}$

$= \frac{AD}{DB}$ ... (1)

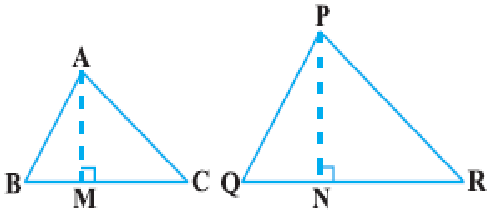
and $\frac{ar(ADE)}{ar(DEC)} = \frac{\frac{1}{2} \times AE \times DM}{\frac{1}{2} \times EC \times DM}$

$= \frac{AE}{EC}$ …(2)

Note that ΔBDE and ΔDEC are on the same base DE and between the same parallels BC and DE.
So, ar(BDE) = ar(DEC)...(3)
Therefore, from (1), (2) and (3), we have:
$\frac{AD}{DB} = \frac{AE}{EC}$

Areas of Similar Triangles:
The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Proof: We are given two triangles ABC and PQR such that ΔABC ~ ΔPQR



We need to prove that

$$\frac{ar(ABC)}{ar(PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$$

For finding the areas of the two triangles, we draw altitudes AM and PN of the triangles.

Now, ar(ABC) = $\frac{1}{2} \times BC \times AM$

and ar(PQR) = $\frac{1}{2} \times QR \times PN$

So, $\frac{ar(ABC)}{ar(PQR)} = \frac{\frac{1}{2} \times BC \times AM}{\frac{1}{2} \times QR \times PN}$ ... (1)

Now, in ΔABM and ΔPQN,

$\angle B = \angle Q$ (As ΔABC ~ ΔPQR)

and $\angle M = \angle N$ (Each is 90°)

So, ΔABM ~ ΔPQN (AA similarity criterion)

Therefore, $\frac{AM}{PN} = \frac{AB}{PQ}$ ... (2)

Also, ΔABC ~ ΔPQR (Given)

So, $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$ ... (3)

Therefore, $\frac{ar(ABC)}{ar(PQR)} = \frac{AB}{PQ} \times \frac{AM}{PN}$

[From (1) & (3)]

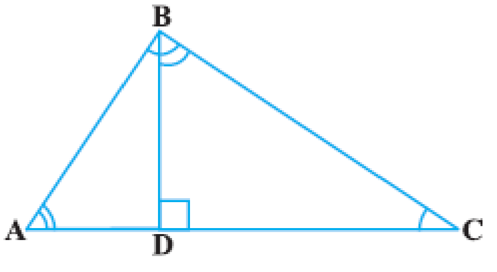
$= \frac{AB}{PQ} \times \frac{AB}{PQ}$ [From (2)]

$= \left(\frac{AB}{PQ}\right)^2$

Now using (3) we get:

$$\frac{ar(ABC)}{ar(PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{CA}{RP}\right)^2$$

Proof of Pythagoras Theorem Using Similarity



You may note that in ΔADB and ΔABC

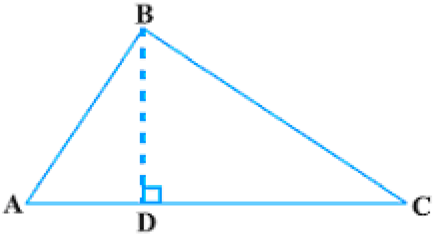
$\angle A = \angle A$

and $\angle ADB = \angle ABC$
So, $\Delta ADB \sim \Delta ABC$
Similarly, $\Delta BDC \sim \Delta ABC$
From (1) and (2), triangles on both sides of the perpendicular BD are similar to the whole triangle ABC.
Also, since $\Delta ADB \sim \Delta ABC$
and $\Delta BDC \sim \Delta ABC$
So, $\Delta BDC \sim \Delta BDC$

If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole triangle and to each other.

Let us now apply this theorem in proving the Pythagoras Theorem:
In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

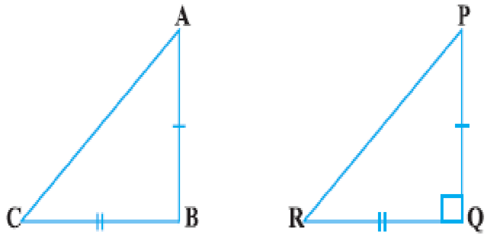
Proof: We are given a right triangle ABC right angled at B.



We need to prove that $AC^2 = AB^2 + BC^2$
Let us draw $BD \perp AC$.
Now, $\Delta ADB \sim \Delta ABC$
So, $\frac{AD}{AB} = \frac{AB}{AC}$ (Sides are proportional)
or $AD \cdot AC = AB^2$ … (1)
Also, $\Delta BDC \sim \Delta ABC$
So, $\frac{CD}{BC} = \frac{BC}{AC}$
or $CD \cdot AC = BC^2$ … (2)
Adding (1) and (2),
$AD \cdot AC + CD \cdot AC = AB^2 + BC^2$
Or $AC (AD + CD) = AB^2 + BC^2$
Or $AC \cdot AC = AB^2 + BC^2$
Or $AC^2 = AB^2 + BC^2$

Converse of Pythagoras Theorem:
In a right triangle, if the square of one side is equal to the sum of the squares of the other two sides, then the angle opposite the first side is a right angle.

Proof: Here we are given a triangle ABC in which $AC^2 = AB^2 + BC^2$
We need to prove that $\angle B = 90°$
To start with, we construct a ΔPQR right angled at Q such that PQ = AB and QR = BC



Now, from ΔPQR, we have:
$PR^2 = PQ^2 + QR^2$ (Pythagoras Theorem, as $\angle Q = 90°$)
Or $PR^2 = AB^2 + BC^2$ (By construction). (1)
But $AC^2 = AB^2 + BC^2$ (Given)… (2)
So, AC = PR [From (1) and (2)] … (3)
Now, in ΔABC and ΔPQR,
AB = PQ (By construction)
BC = QR (By construction)
AC = PR [Proved in (3) above]
So, $\Delta ABC \cong \Delta PQR$ (SSS congruence)
Therefore, $\angle B = \angle Q$ (CPCT)
But $\angle Q = 90°$ (By construction)
So, $\angle Q = 90°$